MA 490: Mathematics Seminar
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From College Mathematics, Vol. 37, No. 2: March 2006,

#825 – Proposed by Juan-Bosco Romero Marquez, Universidad de Valladolia, Valladolia, Spain:

If a > 0, then let

$$J(a) = \int_0^\infty \frac{dx}{(a+x)(1+a^2x^2)}$$

and

$$K(a) = \int_0^\infty \frac{dx}{(1+ax)(a^2+x^2)}$$

Evaluate

$$L = \lim_{a\to 1}\left[\frac{K(a)}{J(a)}\right]^{\frac{1}{a-1}}$$

I. Partial Fractions to simplify integrands.

First,

$$j(a) = \frac{1}{(a+x)(1+a^2x^2)} = \frac{A}{a+x} + \frac{Bx+C}{1+a^2x^2}$$

$$1 = A(1+a^2x^2)+(Bx+C)(a+x)$$

$$1 = A+Aa^2x^2+Bax+Bx^2+Ca+Cx$$

Setting the left side figures equal to the corresponding coefficients, we have:

$$0 = Aa^2+B \qquad 0 = Ba+C \qquad 1 = A+Ca$$

and so we see:

$$B = -Aa^2 \qquad C = -Ba \qquad 1 = A+Ca$$

$$B = -A a^2 \qquad C = -(-A a^2) * a \qquad 1 = A + C a$$
$$B = -A a^2 \qquad C = A a^3 \qquad 1 = A + C a$$
$$B = -A a^2 \qquad C = A a^3 \qquad 1 = A + (A a^3) * a$$
$$B = -A a^2 \qquad C = A a^3 \qquad 1 = A(1 + a^4)$$
$$B = -A a^2 \qquad C = A a^3 \qquad A = \frac{1}{1 + a^4}$$
$$B = -\left(\frac{1}{1 + a^4}\right) * a^2 \qquad C = \left(\frac{1}{1 + a^4}\right) * a^3 \qquad A = \frac{1}{1 + a^4}$$
$$B = \frac{-a^2}{1 + a^4} \qquad C = \frac{a^3}{1 + a^4} \qquad A = \frac{1}{1 + a^4}$$
$$B = \frac{-a^2}{1 + a^4} \qquad C = \frac{a^3}{1 + a^4} \qquad A = \frac{1}{1 + a^4}$$

and so we have

$$j(a) = \frac{1}{(a+x)(1 + a^2 x^2)} = \frac{A}{a + x} + \frac{Bx + C}{1 + a^2 x^2} = \frac{1}{(a+x)(1 + a^4)} - \frac{a^2(x - a)}{(1 + a^2 x^2)(1 + a^4)}$$

Now similarly,

$$k(a) = \frac{1}{(1 + a x)(a^2 + x^2)} = \frac{A}{1 + a x} + \frac{Bx + C}{a^2 + x^2}$$

$$1 = A a^2 + A x^2 + B x + B a x^2 + C + C a x$$

Setting the left side figures equal to the corresponding coefficients, we have:

$$0 = A + B a \qquad 0 = B + C a \qquad 1 = A a^2 + C$$
$$B = \frac{-A}{a} \qquad 0 = B + C a \qquad 1 = A a^2 + C$$
$$B = \frac{-A}{a} \qquad C = \frac{-B}{a} \qquad 1 = A a^2 + C$$
$$B = \frac{-A}{a} \qquad C = \frac{-\left(\frac{-A}{a}\right)}{a} \qquad 1 = A a^2 + C$$
$$B = \frac{-A}{a} \qquad C = \frac{A}{a^2} \qquad 1 = A a^2 + \frac{A}{a^2}$$
$$B = \frac{-A}{a} \qquad C = \frac{A}{a^2} \qquad a^2 = A a^4 + A$$

$$B=\frac{-A}{a} \qquad C=\frac{A}{a^2} \qquad a^2=A(a^4+1)$$

$$B=\frac{-A}{a} \qquad C=\frac{A}{a^2} \qquad A=\frac{a^2}{1+a^4}$$

$$B=\frac{-\left(\frac{a^2}{1+a^4}\right)}{a} \qquad C=\frac{\left(\frac{a^2}{1+a^4}\right)}{a^2} \qquad A=\frac{a^2}{1+a^4}$$

$$B=\frac{-a}{1+a^4} \qquad C=\frac{1}{1+a^4} \qquad A=\frac{a^2}{1+a^4}$$

which gives us

$$k(a) = \frac{1}{(1+ax)(a^2+x^2)} = \frac{A}{1+ax} + \frac{Bx+C}{a^2+x^2} = \frac{a^2}{(1+ax)(1+a^4)} - \frac{ax-1}{(a^2+x^2)(1+a^4)}$$

II. Anti differentiate

First,

$$\int j(a)\,dx = \int \frac{1}{(a+x)(1+a^4)} - \frac{a^2(x-a)}{(1+a^2x^2)(1+a^4)}\,dx$$

$$= \int \frac{dx}{(a+x)(1+a^4)} - \int \frac{a^2(x-a)}{(1+a^2x^2)(1+a^4)}\,dx$$

$$= \frac{1}{1+a^4}*\int \frac{dx}{a+x} - \frac{a^2}{1+a^4}*\int \frac{x-a}{1+a^2x^2}\,dx$$

$$= \frac{1}{1+a^4}*\int \frac{dx}{a+x} - \left[\frac{a^2}{1+a^4}*\int \frac{x}{1+a^2x^2} - \frac{a}{1+a^2x^2}\,dx\right]$$

$$= \frac{1}{1+a^4}*\int \frac{dx}{a+x} - \left[\frac{a^2}{1+a^4}*\left(\int \frac{x}{1+a^2x^2}\,dx - \int \frac{a}{1+a^2x^2}\,dx\right)\right]$$

$$= \frac{1}{1+a^4}*\int \frac{dx}{a+x} - \left[\frac{a^2}{1+a^4}*\left(\int \frac{x}{1+a^2x^2}\,dx - a*\int \frac{dx}{1+a^2x^2}\right)\right]$$

From literature we are given:

$$\int \frac{dx}{b\,x+c} = \frac{1}{b}\ln|b\,x+c| \qquad \text{Int(1)}$$

$$\int \frac{x}{b\,x^2+c}\,dx = \frac{1}{2\mathrm{b}}\ln|b\,x^2+c| \qquad \text{Int(2)}$$

$$\int \frac{dx}{b\,x^2+c} = \frac{1}{\sqrt{b\,c}}\arctan\left(x\sqrt{\frac{b}{c}}\right) \qquad \text{Int(3)}$$

and so, with Int(1), Int (2), and Int(3), we get:

$$\frac{1}{1+a^4}*\frac{1}{1}*\ln|1\,x+a|-\left\{\frac{a^2}{1+a^4}*\left[\frac{1}{2\,a^2}\ln|a^2\,x^2+1|-a*\left(\frac{1}{\sqrt{a^2*1}}\arctan\left(x\sqrt{\frac{a^2}{1}}\right)\right)\right]\right\}$$

$$= \frac{\ln|x+a|}{1+a^4}-\left\{\frac{a^2}{1+a^4}*\left[\frac{\ln|a^2\,x^2+1|}{2\,a^2}-a\left(\frac{1}{a}\arctan(x\,a)\right)\right]\right\}$$

$$= \frac{\ln|x+a|}{1+a^4}-\left[\frac{a^2}{1+a^4}*\left(\frac{\ln|a^2\,x^2+1|}{2\,a^2}-\arctan(x\,a)\right)\right]$$

We can drop the absolute value signs because by definition, a and x are greater than zero.

$$= \frac{\ln(x+a)}{1+a^4} - \left[\frac{\ln(a^2\,x^2+1)}{2(1+a^4)} - \frac{a^2\arctan(a\,x)}{1+a^4}\right]$$

$$= \frac{\ln\left(\frac{(x+a)^2}{a^2\,x^2+1}\right)}{2(1+a^4)} + \frac{a^2\arctan(a\,x)}{1+a^4}$$

Now similarly,

$$\int k(a)\,dx = \int \frac{a^2}{(1+a\,x)(1+a^4)} - \frac{a\,x-1}{(a^2+x^2)(1+a^4)}\,dx$$

$$= \int \frac{a^2}{(1+a\,x)(1+a^4)}\,dx - \int \frac{a\,x-1}{(a^2+x^2)(1+a^4)}\,dx$$

$$= \frac{a^2}{1+a^4} * \int \frac{dx}{1+ax} - \frac{1}{1+a^4} * \int \frac{ax-1}{(a^2+x^2)} dx$$

$$= \frac{a^2}{1+a^4} * \int \frac{dx}{1+ax} - \left[\frac{1}{1+a^4} * \left(a * \int \frac{x}{(a^2+x^2)} dx - \int \frac{dx}{a^2+x^2} \right) \right]$$

and so, with Int(1), Int (2), and Int(3), we get:

$$= \frac{a^2}{1+a^4} * \frac{1}{a} * \ln|ax+1| - \left[\frac{1}{1+a^4} * \left(a * \frac{\ln|x^2+a^2|}{2} - \frac{1}{a} \arctan\left(\frac{x}{a}\right) \right) \right]$$

We can drop the absolute value signs because by definition, a and x are greater than zero.

$$= \frac{a\ln(ax+1)}{1+a^4} - \left[\frac{a\ln(x^2+a^2)}{2(1+a^4)} - \frac{\arctan(\frac{x}{a})}{a(1+a^4)} \right]$$

$$= \frac{a\ln\left(\frac{(ax+1)^2}{x^2+a^2}\right)}{2(1+a^4)} + \frac{\arctan\left(\frac{x}{a}\right)}{a(1+a^4)}$$

III. Evaluate Integrals

$$J(a) = \frac{\ln\left(\frac{(x+a)^2}{a^2x^2+1}\right)}{2(1+a^4)} + \frac{a^2\arctan(ax)}{1+a^4} \Big|_0^\infty$$

$$= \left[\frac{-\ln(a^2)}{2(1+a^4)} + \frac{a^2\pi}{2(1+a^4)} \right] - \left[\frac{\ln(a^2)}{2(1+a^4)} - 0 \right] = \frac{a^2\pi - \ln(a^4)}{2(1+a^4)}$$

$$K(a) = \frac{a\ln\left(\frac{(ax+1)^2}{x^2+a^2}\right)}{2(1+a^4)} + \frac{\arctan\left(\frac{x}{a}\right)}{a(1+a^4)} \Big|_0^\infty$$

$$= \left[\frac{a\ln(a^2)}{2(1+a^4)} + \frac{\pi}{2a(1+a^4)}\right] - \left[\frac{-a\ln(a^2)}{2(1+a^4)} + 0\right] = \frac{a^2\ln(a^4)+\pi}{2a(1+a^4)}$$

IV. Simplify Quotient

$$\frac{K(a)}{J(a)} = \frac{\left(\dfrac{a^2\ln(a^4)+\pi}{2a(1+a^4)}\right)}{\left(\dfrac{a^3\pi-a\ln(a^4)}{2a(1+a^4)}\right)} = \frac{a^2\ln(a^4)+\pi}{a^3\pi-a\ln(a^4)}$$

V. Set $1+\frac{1}{kn}$ equal to the above quotient, substitute $1+\frac{1}{n}$ for 'a', and solve for k.

$$1+\frac{1}{kn} = \frac{a^2\ln(a^4)+\pi}{a^3\pi-a\ln(a^4)} = \frac{\pi+\left(1+\frac{1}{n}\right)^2\ln\left(\left(1+\frac{1}{n}\right)^4\right)}{\left(1+\frac{1}{n}\right)^3\pi-\left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^4\right)}$$

So,

$$\frac{1}{k} = \frac{n\left(\pi+\left(1+\frac{1}{n}\right)^2\ln\left(\left(1+\frac{1}{n}\right)^4\right)-\left(1+\frac{1}{n}\right)^3\pi+\left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^4\right)\right)}{\left(1+\frac{1}{n}\right)^3\pi-\left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^4\right)}$$

and thus,

$$k = \frac{\left(1+\frac{1}{n}\right)^3\pi-\left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^4\right)}{n\left(\pi+\left(1+\frac{1}{n}\right)^2\ln\left(\left(1+\frac{1}{n}\right)^4\right)-\left(1+\frac{1}{n}\right)^3\pi+\left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^4\right)\right)}$$

$$= \frac{\left(1+\frac{1}{n}\right)^{3}\pi - \left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^{4}\right)}{n\pi + n*\left(1+\frac{1}{n}\right)^{2}\ln\left(\left(1+\frac{1}{n}\right)^{4}\right) - \left(1+\frac{1}{n}\right)^{3} n\pi + n*\left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^{4}\right)}$$

$$= \frac{\left(1+\frac{1}{n}\right)^{3}\pi - \left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^{4}\right)}{n\pi - \left(1+\frac{1}{n}\right)^{3} n\pi + \left(1+\frac{1}{n}\right)^{2}\ln\left(\left(1+\frac{1}{n}\right)^{4n}\right) + \left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^{4n}\right)}$$

VI. Evaluate k as $a \to 1$ from the right, or as $n \to \infty$

From literature we are given

$$\lim_{n\to\infty}\left(1+\frac{1}{nj}\right)^{n} = e^{\frac{1}{j}} \qquad \text{Lim(1)}$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{ni} = e^{i} \qquad \text{Lim(2)}$$

and can so construct

$$\lim_{n\to\infty}\left(1+\frac{1}{nj}\right)^{nih} = e^{\frac{ih}{j}} \qquad \text{Lim(3)}$$

thus,

$$\lim_{n\to\infty} k = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)^{3}\pi - \left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^{4}\right)}{n\pi - \left(1+\frac{1}{n}\right)^{3} n\pi + \left(1+\frac{1}{n}\right)^{2}\ln\left(\left(1+\frac{1}{n}\right)^{4n}\right) + \left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^{4n}\right)}$$

$$= \lim_{n\to\infty} \frac{1*\pi-1*\ln(1)}{n\pi-\left(1+\frac{1}{n}\right)^3 n\pi+1*\ln(e^4)+1*\ln(e^4)}$$ [using Lim(2) for i=4]

$$= \lim_{n\to\infty} \frac{\pi-1*0}{n\pi-\left(1+\frac{1}{n}\right)^3 n\pi+4+4}$$

Now we note

$$n\pi-\left(1+\frac{1}{n}\right)^3 n\pi=n\pi-\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)n\pi=n\pi-\left(n\pi+3\pi+\frac{3\pi}{n}+\frac{\pi}{n^2}\right)=-3\pi-\frac{3\pi}{n}-\frac{\pi}{n^2}$$

and

$$\lim_{n\to\infty} -3\pi-\frac{3\pi}{n}-\frac{\pi}{n^2}=-3-0-0\pi=-3\pi$$

thus

$$\lim_{n\to\infty} \frac{\pi-1*0}{n\pi-\left(1+\frac{1}{n}\right)^3 n\pi+4+4}=\frac{\pi}{-3\pi+4+4}=\frac{\pi}{8-3\pi}$$

VII. Evaluate k as $a\to 1$ **from the left, or as** $n\to-\infty$

$$\lim_{n\to-\infty} k = \lim_{n\to-\infty} \frac{\left(1+\frac{1}{n}\right)^3\pi-\left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^4\right)}{n\pi-\left(1+\frac{1}{n}\right)^3 n\pi+\left(1+\frac{1}{n}\right)^2\ln\left(\left(1+\frac{1}{n}\right)^{4n}\right)+\left(1+\frac{1}{n}\right)\ln\left(\left(1+\frac{1}{n}\right)^{4n}\right)}$$

$$= \lim_{n\to-\infty} \frac{1*\pi-1*\ln(1)}{n\pi-\left(1+\frac{1}{n}\right)^3 n\pi+1*\ln\left(e^{\frac{-4}{-1}}\right)+1*\ln\left(e^{\frac{-4}{-1}}\right)}$$ [using Lim(3) for h=-1, i=4 and j=-1]

$$= \lim_{n\to-\infty} \frac{\pi-1*0}{n\pi-\left(1+\frac{1}{n}\right)^3 n\pi+4+4}$$

$$= \lim_{n\to-\infty} \frac{\pi-1*0}{-3\pi-\frac{3\pi}{n}-\frac{\pi}{n^2}+4+4}=\frac{\pi}{-3\pi+4+4}=\frac{\pi}{8-3\pi}$$

We now note that the above limits evaluate similarly whether from the left or the right.

VIII. Evaluate L

Since

$$\lim_{n\to\pm\infty} k=\frac{\pi}{8-3\pi}$$

we have

$$L = \lim_{a\to1}\left[\frac{K(a)}{J(a)}\right]^{\frac{1}{a-1}} = \lim_{a\to1}\left[\frac{a^2\ln(a^4)+\pi}{a^3\pi-a\ln(a^4)}\right]^{\frac{1}{a-1}} = \lim_{n\to\pm\infty}\left(1+\frac{1}{nk}\right)^n = e^{\frac{1}{k}}$$

and so finally,

$$L = e^{\frac{8-3\pi}{\pi}}$$